Stability of equilibrium points:

Consider a dynamical system described by $\dot{x} = F(x)$. Let x^e be an equilibrium pt. Then, starting from $x(0) = x^e$, the system will stay at x^e forever. However, if you perturb the system from x^e, it can have different behaviors:

(I) The system goes back to x^e from all possible "small" perturbations.

(II) The system moves away from x^e for some "small" perturbation.

(III) The system remains "close" to x^e for all "small" perturbations, but may not tend toward x^e asymptotically.

Agenda:

① Study stability of the origin of a linear dynamical system of the form $\dot{X} = AX$.

② Study stability of an eq. pt. X^e of a general dynamical system $\dot{X} = F(X)$ through linearization of F around X^e.

Stability of origin of $\dot{X} = AX$.

- To motivate the study, consider a scalar dynamical system of the form $\dot{x} = ax$. Then, the origin $(x = 0)$ is an equilibrium pt.

<u>Case I: $a > 0$</u>. Then, if you perturb the system to $\varepsilon > 0$, you have $\dot{x} = a\varepsilon > 0$, i.e., a positive velocity. As a result, the system will tend to move away from $x = 0$.

Convince yourself that if you started from $x(0) = \varepsilon < 0$, the system will again move away from $x = 0$. Therefore, $x = 0$ is unstable.

<u>Case II : $a < 0$.</u> If you start from $x(0) = \varepsilon > 0$, then $\dot{x}(0) = a\varepsilon < 0$, i.e., it will have a negative velocity, and the system will move towards the origin. Convince yourself that the system will tend to the origin, even if you started at $x(0) = \varepsilon < 0$. Therefore, $x = 0$ is stable.

Takeaway from this example: Stability of the origin for a scalar dynamical system $\dot{x} = ax$ only depends on the sign of a.

$a > 0 \Rightarrow$ unstable.
$a < 0 \Rightarrow$ stable.

- For general linear time-invariant systems $\dot{x} = Ax$, the stability of the origin is determined as follows:

  Let $\lambda_1, \ldots, \lambda_n$ be the complex eigenvalues of $A \in \mathbb{R}^{n \times n}$. Then,

  I. if $\mathrm{Re}\{\lambda_i\} < 0$ for all $i = 1, \ldots, n$, then the origin is
  table.

  II. if $\mathrm{Re}\{\lambda_i\} > 0$ for any $i = 1, \ldots, n$, then the origin is unstable.

  III. if $\mathrm{Re}\{\lambda_i\} \leq 0$ for all $i = 1, \ldots, n$, and $\mathrm{Re}\{\lambda_i\} = 0$ for some $i = 1, \ldots, n$, then the origin is marginally stable.

Here, $\mathrm{Re}\{\cdot\}$ denotes the real part of the complex number.

Remark: For linear dynamical systems, we often refer to the "system" being stable or not, rather than the origin.

<u>Example</u>: Consider a 2^{nd}-order dynamical system $\dot{X} = AX$, where $A = \begin{pmatrix} 7 & -1 \\ -2 & 1 \end{pmatrix}$. State whether the system is stable, unstable, or marginally stable.

- Calculating eigenvalues of A: Solve for λ's that satisfy $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{pmatrix} 7-\lambda & -1 \\ -2 & 1-\lambda \end{pmatrix}$$

$$= (7-\lambda)(1-\lambda) - (-2)(-1)$$

$$= 7 - \lambda - 7\lambda + \lambda^2 - 2$$

$$= \lambda^2 - 8\lambda + 5.$$

Solving $\lambda^2 - 8\lambda + 5 = 0$, we get,

$$\lambda = \frac{8 \pm \sqrt{64-20}}{2} = 4 \pm \sqrt{11}.$$

$4 + \sqrt{11} > 0 \Rightarrow$ the system is unstable.

Example: Consider a dynamical system described by $\ddot{x} + 2\xi\dot{x} + \omega_0^2 x = 0$, where $\xi \geqslant 0$ and ω_0 are constants. State when the system is stable, unstable, or marginally stable.

- Convert ODE description to state-space form:

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where $x_1 = x$, $x_2 = \dot{x}$.

Then,
$$\dot{X} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix}$$
$$= \begin{pmatrix} \dot{x} \\ -2\xi\dot{x} - \omega_0^2 x \end{pmatrix}$$
$$= \begin{pmatrix} x_2 \\ -2\xi x_2 - \omega_0^2 x_1 \end{pmatrix}$$
$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\xi \end{pmatrix}}_{:= A} X.$$

- Compute eigenvalues of A:

$$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -\omega_0^2 & -2\xi-\lambda \end{pmatrix}$$

$$= (-\lambda)(-2\xi-\lambda) - 1(-\omega_0^2)$$

$$= \lambda^2 + 2\xi\lambda + \omega_0^2.$$

Setting $\det(A-\lambda I)=0$, we get,

$$\lambda = \frac{-2\xi \pm \sqrt{4\xi^2 - 4\omega_0^2}}{2}$$

$$= -\xi \pm \sqrt{\xi^2-\omega_0^2}.$$

Case 1: $\omega_0 \neq 0$. Then, three cases can arise:

- $\xi^2 - \omega_0^2 > 0$

  $\xi^2-\omega_0^2 < \xi^2 \Rightarrow -\xi + \sqrt{\xi^2-\omega_0^2} < 0$.

  Also, $-\xi - \sqrt{\xi^2-\omega_0^2} < 0 \Rightarrow$ system is stable.

- $\xi^2 - \omega_0^2 = 0$

  $\lambda = -\xi, -\xi \Rightarrow$ system is stable.

- $\xi^2 - \omega_0^2 < 0$

$$\lambda = -\xi \pm j\sqrt{\omega_0^2 - \xi^2}.$$

$\Rightarrow \mathrm{Re}\{\lambda\} = -\xi, -\xi < 0.$

$\Rightarrow$ system is stable.

Case 2: $\omega_0 = 0$. Then, $\lambda = -2\xi, 0$.

$\Rightarrow$ system is marginally stable.

Stability of eq. pt. of a possibly non-linear dynamical system X^e.

So far, we have studied the stability of the origin of a linear dynamical system of the form $\dot{X} = AX$. Now, let's study the stability of an eq. pt. X^e of a possibly non-linear system $\dot{x} = F(x)$.

Steps involved:

- Linearize F around X^e.
- Study the stability of the linearized system.
- Infer about stability of eq. pt. using Lyapunov's theorem.

- Linearizing F around x^e:

Since x^e is an eq. pt., $F(x^e) = 0$.

Define $y = x - x^e$.

Then,
$$
\begin{aligned}
\dot{y} &= \dot{x} \\
&= F(x) \\
&= F(x^e + y) \\
&\approx F(x^e) + \nabla F(x^e) \cdot y \\
&= \nabla F(x^e) \cdot y
\end{aligned}
$$

for y close to 0, where $\nabla F(x^e)$ is the Jacobian of F evaluated at x^e.

The local "linearized" dynamical system around x^e is given by

$$\dot{y} = \underbrace{\nabla F(x^e)}_{:= A} \cdot y = A \cdot y.$$

Calculating $\nabla F(X^e)$: If $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, then express $F(X)$ as

$$F(X) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{pmatrix}.$$

$$\Rightarrow \nabla F(X) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

Get $\nabla F(X^e)$ by evaluating the derivatives at X^e.

Example: Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\dot{X} = \begin{pmatrix} -x_1 + x_2^2 \\ x_1 - 1 \end{pmatrix}$.

$X^e = \begin{pmatrix} x_1^e \\ x_2^e \end{pmatrix}$ satisfies $-x_1^e + (x_2^e)^2 = 0$, $x_1^e = 1$.

$$\Rightarrow X^e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$F(x) = \begin{pmatrix} -x_1 + x_2^2 \\ x_1 - 1 \end{pmatrix}$$

$$\Rightarrow \nabla F(x) = \begin{pmatrix} \frac{\partial}{\partial x_1}\left(-x_1 + x_2^2\right) & \frac{\partial}{\partial x_2}\left(-x_1 + x_2^2\right) \\ \frac{\partial}{\partial x_1}(x_1 - 1) & \frac{\partial}{\partial x_2}(x_1 - 1) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & +2x_2 \\ 1 & 0 \end{pmatrix}.$$

Linearized system around $x^e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$$\dot{y} = \nabla F\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot y = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} \cdot y$$

Linearized system around $x^e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$:

$$\dot{y} = \nabla F\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot y = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix} \cdot y.$$

- Study the stability of the linearized systems around each eq. pt:

  Steps: Utilize the recipe for analyzing stability of the linearized system $\dot{y} = \nabla F(x^e) \cdot y$ by computing the eigenvalues of $\nabla F(x^e)$.

Back to our example:

- Linearized system around $x^e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $\dot{y} = \underbrace{\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}}_{:= A} \cdot y$

$$\det(A - \lambda I) = (-1-\lambda)(-\lambda) - 2 \cdot 1$$
$$= \lambda^2 + \lambda - 2$$

Setting it to zero, we get $\lambda = \dfrac{-1 \pm \sqrt{1+8}}{2}$

$$= \frac{-1 \pm 3}{2} = 1, -2.$$

One of the eigenvalues is +ve

$\Rightarrow$ The linearized system around $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is unstable.

Linearized system around $x^e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$: $\dot{y} = \underbrace{\begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}}_{:= A} \cdot y$.

$$\det(A - \lambda I) = (-1-\lambda)(-\lambda) - 1(-2)$$
$$= \lambda^2 + \lambda + 2$$

Eigenvalues are given by $\lambda = \dfrac{-1 \pm \sqrt{1-8}}{2}$

$$= \frac{-1 \pm j\sqrt{7}}{2}$$

$\Rightarrow \mathrm{Re}\{\lambda\} = -\frac{1}{2}, -\frac{1}{2} < 0$

$\Rightarrow$ Linearized system around $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is stable.

So far, we have only deduced the stability of the "linearized" systems ~~the~~ around eq. pt. Now, let's deduce the stability of the eq. pts of the non-linear system $\dot{x} = F(x)$ using Lyapunov's theorem.

Lyapunov's theorem: Consider a dynamical system $\dot{x} = F(x)$, and let x^e be an equilibrium point. Then:

- If the linearized system around x^e is stable, then x^e is asymptotically stable for the system $\dot{x} = F(x)$.
- If the linearized system around x^e is unstable, then x^e is unstable for the system $\dot{x} = F(x)$.

Remark 1: Asymptotically stable means that $x(t) \to x^e$, if $x(0)$ is "close enough" to x^e.

Remark 2: If the linearized system is marginally stable, then NOTHING can be deduced about the stability of the eq. pt. of $\dot{x} = F(x)$.

Back to our example :

Recall that our example system was given by $\dot{X} = \begin{pmatrix} -x_1 + x_2^2 \\ x_1 - 1 \end{pmatrix}$, where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$X^e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

" Linearized system around $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is $\dot{Y} = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} \cdot Y$ that is unstable.

" Linearized system around $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is $\dot{Y} = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix} \cdot Y$ is stable.

From Lyapunov's theorem, we get

- $X^e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is unstable, meaning a "small" perturbation from $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ can make $X(t)$ diverge from $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- $X^e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is asymptotically stable, meaning after any "small" perturbation from $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $X(t)$ will approach $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as $t \to \infty$.